MATH 2250 Calculus
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Name: Solutions
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Test No. 4

You have 75 minutes to complete the **seven** problems on this exam. No calculator is allowed, nor are any materials aside from a pen or pencil allowed. Follow all instructions properly for full credit, and assume that you are required to show your work unless otherwise specified.

Problem 1 (15 points) Use $n = 2$ iterations of Newton's method with $x_0 = 0$ to find an approximate solution to $x^2 + x = 3$.

$f(x) = x^2 + x - 3$

$f'(x) = 2x + 1$

$$x_1 = 0 - \frac{f(0)}{f'(0)} = \frac{-(-3)}{1} = 3$$

$$x_2 = 3 - \frac{3^2 + 3 - 3}{2(3) + 1} = 3 - \frac{9}{7} = \frac{12}{7}$$

Problem 2 (25 points) Find the following antiderivatives, not forgetting the arbitrary constant C.

a $\int \frac{1}{x} + 3e^{2x} dx$ $= \log(|x|) + 3\left(\frac{e^{2x}}{2}\right) + C$

$= \log(|x|) + \frac{3}{2}e^{2x} + C$

b $\int \cos(5x) dx$ $= \frac{1}{5}\sin(5x) + C$

c $\int 2x^3 + 3x^2 dx$ $= 2\left(\frac{x^4}{4}\right) + 3\left(\frac{x^3}{3}\right) + C$

$= \frac{1}{2}x^4 + x^3 + C$

d $\int 5 - \frac{1}{\sqrt[3]{x}} dx$ $= 5x - \frac{x^{-1/3+1}}{-1/3+1} + C$

$= 5x + \frac{3}{2}x^{2/3} + C$

e $\int \sec^2(x) + \frac{1}{1+x^2} dx$ $= \tan(x) + \arctan(x) + C$

Problem 3 (10 points) A particle moves on a coordinate line with acceleration $a(t) = \frac{d^2s}{dt^2} = 15\sqrt{t} - 3t^{-1/2}$, subject to the initial conditions $s'(1) = 4$ and $s(1) = 0$.

a Find the velecity function $v(t) = \frac{ds}{dt}$.

$$v(t) = \int a(t)\,dt = \int 15t^{1/2} - 3t^{-1/2}\,dt = 15\frac{t^{3/2}}{3/2} - 3\frac{t^{1/2}}{1/2} + C$$

$$= 10t^{3/2} - 6t^{1/2} + C$$

$$v(1) = 4 \Rightarrow \quad v(1) = 10 - 6 + C = 4 \Rightarrow \quad C = 0$$

$$\Rightarrow \quad v(t) = 10t^{3/2} - 6t^{1/2}$$

b Find the position function $s(t)$.

$$s(t) = \int v(t)\,dt = \int 10t^{3/2} - 6t^{1/2}\,dt$$

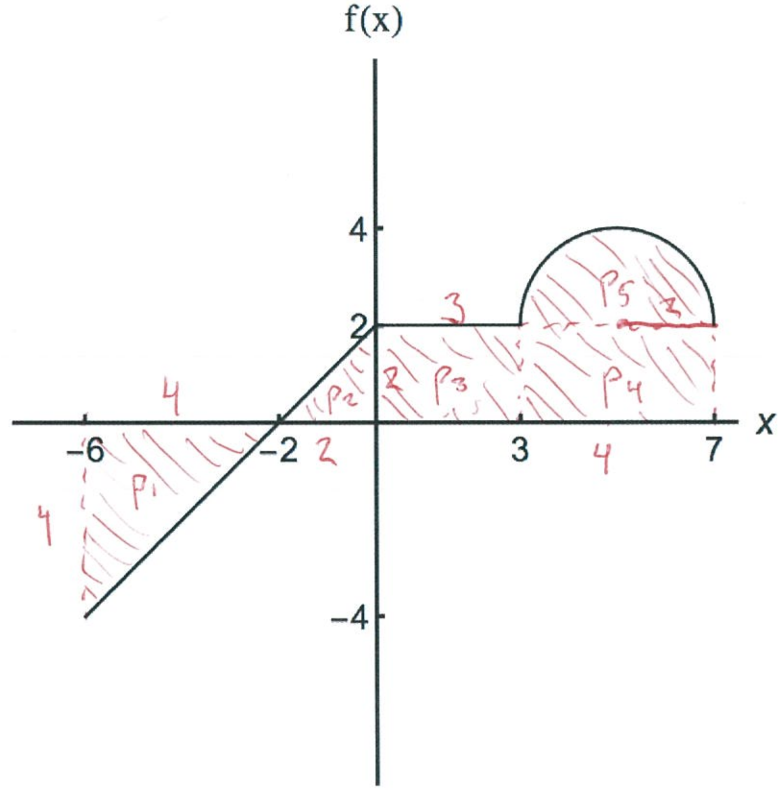
$$= 10\frac{t^{5/2}}{5/2} - 6\frac{t^{3/2}}{3/2} + C$$

$$= 4t^{5/2} - 4t^{3/2} + C$$

$$s(1) = 0 \Rightarrow \quad 4 - 4 + C = 0 \Rightarrow \quad C = 0$$

$$\Rightarrow \quad s(t) = 4t^{5/2} - 4t^{3/2}$$

Problem 4 (10 points) Find the definite integral $\int_{-6}^{7} f(x)dx$ using the graph of $f(x)$ given below. Show as much work as you can for partial credit. (The portion of the graph that looks like a semicircle is in fact a semicircle).



$$\int_{-6}^{7} f(x)\,dx = -P_1 + P_2 + P_3 + P_4 + P_5$$

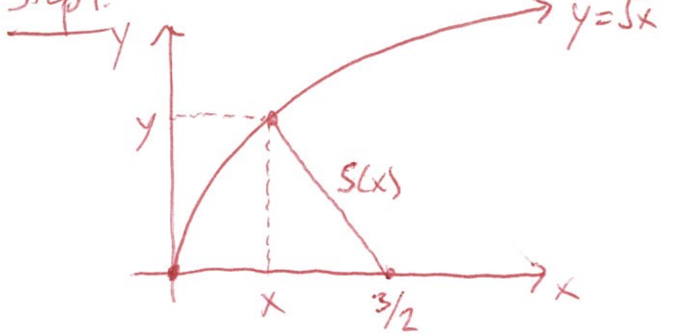
$$= -\frac{1}{2}4^2 + \frac{1}{2}2^2 + 6 + 8 + \frac{1}{2}\pi 2^2$$

$$= -8 + 2 + 14 + 2\pi$$

$$= 8 + 2\pi$$

Problem 5 (20 points) How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? (*Hint: you can minimize the square of the distance to avoid square roots but get the same answer.*)

Step 1:



Step 2: $S^2(x) = (x - 3/2)^2 + y^2$

Step 3: $y = \sqrt{x}$

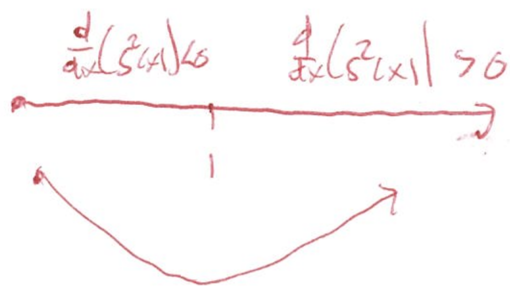
$$\Rightarrow S^2(x) = (x - 3/2)^2 + (\sqrt{x})^2$$

$$= (x - 3/2)^2 + x$$

Step 4: $\operatorname{dom}(s^2) = [0, \infty)$

Step 5: $\frac{d}{dx}(s^2(x)) = \frac{d}{dx}((x - 3/2)^2 + x) = 2(x - 3/2) + 1$

always exits $\Rightarrow$ $0 \overset{\text{Set}}{=} 2x - 3 + 1 \Rightarrow 2 = 2x \Rightarrow x = 1$



Step 6:

So the minimum occurs at $x = 1$, and the closest the curve comes to the point is $s(1) = \sqrt{(1 - 3/2)^2 + 1}$

Problem 6 (20 points) For this problem, you will need the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

a First use either high school geometry or the Fundamental Theorem of Calculus to compute the definite integral $\int_0^1 3x + 1dx$.

$$\int_0^1 3x+1\,dx = \left[3\frac{x^2}{2} + x\right]_0^1 = \frac{3}{2} + 1 = \frac{5}{2}$$

b Using the definition and picking c_k as the right end point of the k^{th} interval, write an expression for the Riemann sum in terms of n, the number of rectangles the interval $[0, 1]$ is divided up into.

i^{th} subinterval: $\left[a + (i-1)\left(\frac{b-a}{n}\right),\ a + i\left(\frac{b-a}{n}\right)\right] = \left[\frac{i-1}{n}, \frac{i}{n}\right]$

$$\Rightarrow c_i = \frac{i}{n} \Rightarrow S(n) = \sum_{i=1}^{n} f(c_i)\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(3\left(\frac{i}{n}\right)+1\right)$$

$$= \frac{1}{n}\left[\frac{3}{n}\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right] = \frac{1}{n}\left[\frac{3}{n}\frac{n(n+1)}{2} + n\right] = \frac{3}{2}\left(\frac{n+1}{n}\right) + 1$$

$$= \frac{3}{2}\left(1 + \frac{1}{n}\right) + 1 = \frac{5}{2} + \frac{3}{2}\left(\frac{1}{n}\right)$$

c Take the limit of the expression from part b to find the definite integral $\int_0^1 3x + 1dx$ using the definition and picking c_k as the right end point of the k^{th} interval.

$$\lim_{n\to\infty}\left(\frac{5}{2}+\frac{3}{2}\left(\frac{1}{n}\right)\right) = \frac{5}{2} = \int_0^1 3x+1\,dx$$

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Problem Bonus (5 points) Why does minimizing the square of the distance give the same answer as minimizing the distance from problem 5? Your answer should contain some math, not just a verbal explanation.

If a function f has an $\text{argmin}(f) = x^*$, then $f(x^*) \leq f(x)$ for all x. Then

$$(f(x^*))^2 \leq (f(x))^2 \text{ if } f(x) \geq 0 \text{ for all } x,$$

because $g(x) = x^2$ is increasing on $[0, \infty)$